# Uitwerking Tentamen Analyse 4 november 2013

1. (a) Take $x_n \geq x_m > 0$. Then $f$ is continuous on any interval $[x_m, x_n]$ (**1 pt.**).
Application of Mean Value Theorem on $[x_m, x_n]$ yields the existence of $c_{nm} \in (x_n, x_m)$ such that

$$|f(x_n) - f(x_m)| \leq f'(c_{nm})||x_n - x_m| \leq M|x_n - x_m|$$

(**3 pt.**)
Let $\epsilon > 0$. Since $(x_n)$ is a Cauchy sequence there exists $N$ such that for all $n, m \geq N, |x_n - x_m| < \frac{\epsilon}{M}$. Hence for all such $n, m$

$$|f(x_n) - f(x_m)| \leq M|x_n - x_m| < \epsilon$$

This proves that $(f(x_n)$ is a Cauchy sequence. (**3 pt.**).
Since any Cauchy sequence converges, $\lim f(x_n)$ exists. (**1 pt.**).

(b) $\lim_{x \to 0^+} f(x)$ exists if and only if $\lim_{n \to \infty} f(x_n)$ exists for all $(x_n)$ with $x_n > 0$ and $\lim x_n = 0$. (**3 pt.**)
Hence by part (a) $\lim_{x \to 0^+} f(x)$ exists. (**1 pt.**).

2. (a) For every $c > 0$ the set $[0, c]$ is compact (bounded and closed). (**2 pt.**).
By a theorem in the book every continuous function on a compact interval is uniformly continuous. (**3 pt.**).

(b) A function $f : [0, \infty) \to \mathbb{R}$is not uniformly continuous if and only if there exists an $\epsilon_0 > 0$ and two sequences $(x_n), (y_n) \in [0, \infty)$ such that

$$|x_n - y_n| \to 0, \quad \text{but } |f(x_n) - f(y_n)| \geq \epsilon_0$$

(**4 pt.**)

(c) Take $x_n = n, y_n = n + \frac{1}{n}$. Then clearly $|x_n - y_n| \to 0$. (**2 pt.**).
Furthermore

$$|f(x_n) - f(y_n)| = |n^2 - (n + \frac{1}{n})^2| = | - 2 - \frac{1}{n^2}| \geq 2$$

(**4 pt.**)
Thus we may e.g. take $\epsilon_0 = 2$ and apply the Sequential Criterion for Nonuniform Continuity. (**2 pt.**)

(d) Suppose that $f$ is *not* uniformly continuous. Then by the above Sequential Criterion for Nonuniform Continuity there exists $\epsilon_0 > 0$ and two sequences $(x_n), (y_n) \in [0, \infty)$ such that

$$|x_n - y_n| \to 0, \quad \text{but } |f(x_n) - f(y_n)| \geq \epsilon_0$$

Then both sequences $(x_n), (y_n)$ should converge to $\infty$ since $f$ is by part (a) uniformly continuous on any interval $[0, c]$ (**4 pt.**)
This means that $\lim f(x_n) = \lim f(y_n) = L$ where $L = \lim_{x \to \infty} f(x)$. But this yields a contradiction with $|f(x_n) - f(y_n)| \geq \epsilon_0 > 0$. (**4 pt.**)

3. (a) The function $g(x) := f(x) - x^2$ is continuous on $[0,1]$, while $g(0) = f(0) \geq 0$ and $g(1) = f(1) - 1 \leq 0$. (**3 pt.**)
If $g(0) = 0$ or $g(1)$ is equal to 0 then we are done (take $x = 0$ or $x = 1$). (**2 pt.**)
Else $g(0) < 0, g(1) > 0$, and thus we may apply the Intermediate Value Theorem to $g$ with intermediate value $L = 0$. This yields the existence of an $x \in (0,1)$ such that $g(x) = 0$ or equivalently $f(x) = x^2$. (**3 pt.**)

   (b) $g(0) = f(0) - 0 = 0$ and $g(1) = f(1) - 1^2 = 0$. Furthermore, $g$ is differentiable (and thus continuous) on $[0,1]$. (**3 pt.**)
Hence we may apply Rolle's theorem to conclude the existence of an $x \in (0,1)$ such that $0 = g'(x) = f'(x) - 2x$. (**5 pt.**)

4. (a) $s_n(x) = \frac{1-(-x)^n}{1-(-x)}(1-x)$ (**2 pt.**), and thus $s_n(x) \to s(x) = \frac{1-x}{1+x}$ for $x < 1$. (**2 pt.**)
Furthermore for $x = 1$ we have $s_n(1) = 0$ and thus convergence to $s(1) = 0$ (**1 pt.**)

   (b)

$$|s_n(x) - s(x)| = |-\frac{1-x}{1+x}(-x)^n| \leq (1-x)x^n \leq \delta^n$$

whenever $x \leq \delta < 1$. (**6 pt.**)
Given $\epsilon < 1$ take $N > \frac{\ln \epsilon}{\ln \delta}$. Then for all $n \geq N$ and for all $x \in [0, \delta]$ we have $|s_n(x) - s(x)| < \epsilon$, and thus uniform convergence. (**2 pt.**)
(**N.B.** An alternative proof is via the Weierstrass M-test: $|(-1)^n x^n (1-x)| \leq \delta^n$.)

   (c) The differential of the function $(1-x)x^n = x^n - x^{n+1}$ is equal to $nx^{n-1} - (n+1)x^n$, and thus the function attains its maximum at $x = \frac{n}{n+1}$ with value

$$(1 - \frac{n}{n+1}) \left(\frac{n}{n+1}\right)^n < \frac{1}{n+1}$$

(**5 pt.**). Hence convergence is uniform (choose $N + 1 > \frac{1}{\epsilon}$). (**2 pt.**)

   (d) The sequence of absolute values becomes $\sum_{n=0}^{\infty} x^n(1-x)$. ((**1 pt.**)
Partial sum in this case is $s_n(x) = (1-x)\frac{1-x^n}{1-x} = 1 - x^n$ for $x < 1$ while $s_n(1) = 0$, and hence the pointwise limit is the function $s(x) = 1, x < 1$ and $s(1) = 0$. (**4 pt.**)
Since this limit function is not continuous, the convergence is *not* uniform. (**2 pt.**)

5. For any partition $P$ we have $U(f,P) \leq U(g,P)$. (**4 pt.**)
Hence $U(f) = \inf_P U(f,P)$ and $U(g) = \inf_P U(g,P)$ satisfies $U(f) \leq U(g)$. Furthermore $\int f = U(f)$ and $\int g = U(g)$.(**6 pt.**)
(Of course, same arguments can be given for $L(f,P), L(g,P)$ and $L(f), L(g)$.)